Solitons in the Calogero-Sutherland Collective-Field Model

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Abstract

In the Bogomol'nyi limit of the Calogero-Sutherland collective-field model we find static-soliton solutions. The solutions of the equations of motion are moving solitons, having no static limit for $\lambda > 1$. They describe holes and lumps, depending on the value of the statistical parametar λ.

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The properties of the Calogero-Sutherland model [1,2], which describes particles in one dimension interacting with a two-body inverse-square potential, are a subject of increasing interest. The Calogero-Sutherland model (CSM) is integrable in terms of symmetric functions, thus representing a valuable specific model to discuss fractional statistics. Similarities between the CSM and the two-dimensional (fractional) quantum Hall system (the Jastrow-type ground-state wave function) indicate that the CSM is linked to an anyonic system as the one-dimensional reduction of the former [3].

The CSM also appears when two-dimensional QCD is formulated as a random matrix model in the large-N limit, restricted to a singlet subspace [4]. For a one-dimensional free fermionic Fock space, which is described by a hermitian matrix model [5], the collective-field Hamiltonian can be introduced [6]. The derived effective lagrangian, having the same dispersion relation as the theory we started from, turns out to be of the Calogero type with the statistical parametar $\lambda = 1/2$ [6]. This can be seen when the collective-field formulation is written for CSM, as was done in ref. [4,7]. The hole excitations in the spectrum are represented as a soliton of the Calogero-Sutherland collective-field theory.

In this Letter we investigate solitons in the CSM in terms of the collective-field Hamiltonian description formulated in ref. [4,7]. We briefly mention the results of ref. [4], and a similar procedure can be applied to the trigonometric interaction [2,4].

The Calogero Hamiltonian, which describes a system of N non-relativistic particles on a line interacting via the two-body inverse-square potential, is given by

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{\lambda(\lambda-1)}{2}\sum_{i\neq j}^{N}\frac{1}{(x_i-x_j)^2}. \tag{1}$$

The dimensionless coupling constant $\lambda(\lambda-1)$ is a positive real number which specifies the statistics of this model. Because of the singularity of the Hamiltonian for $x_i = x_j$, the wave function ought to have a prefactor which will vanish for coincident particles. We extract this prefactor in the form

$$\Psi(x_1, x_2, \ldots, x_N) = \prod_{i<j}^{N}(x_i - x_j)^{\lambda}\Phi(x_1, x_2, \ldots, x_N), \tag{2}$$

and obtain the new Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{d^2}{dx_i^2} - \lambda\sum_{i\neq j}^{N}\frac{1}{x_i - x_j}\frac{d}{dx_i}, \tag{3}$$

acting on the residual wave function Φ. The Hamiltonian (3) is now suitable for transformation into a collective-field representation. It has been shown that, in the large-N limit, the Hamiltonian can be expressed entirely in terms of the density of particles $\rho(x)$ and its canonical conjugate $\pi(x) = -i\frac{\delta}{\delta\rho(x)}$. The Jacobian of the transformation from x_i into $\rho(x)$ rescales the wave functional

$$\Phi(x_1, x_2, \ldots, x_N) = J^{1/2}\Phi(\rho), \tag{4}$$

resulting in the hermitian collective-field Hamiltonian

$$\begin{aligned} H = \frac{1}{2}\int dx\rho(x)(\partial_x\pi)^2 + \frac{1}{8}\int dx\rho(x)\left(\partial_x\frac{\ln J}{\delta\rho(x)}\right)^2 - \\ -\frac{1}{4}\int dx\frac{\delta\omega(x)}{\delta\rho(x)}, \end{aligned} \tag{5}$$

with

$$\omega(x) = (\lambda-1)\partial_x^2\rho(x) + 2\lambda\partial_x\left(\rho(x)⨍dy\frac{\rho(y)}{x-y}\right), \tag{6}$$

and the Jacobian determined from the hermicity condition

$$\partial_x\left(\rho(x)\partial_x\frac{\ln J}{\delta\rho(x)}\right) = \omega(x). \tag{7}$$

The last singular term plays the role of the counter term, and does not give a contribution in the leading order in N. To find the ground-state energy of our system, we assume that the corresponding collective-field configuration is static and has a vanishing momentum π. Therefore, the leading part of the collective-field Hamiltonian in the $1/N$ expansion is given by the effective potential

$$V_{eff}(\rho) = \frac{1}{8}\int dx\rho(x)\left(\partial_x\frac{\ln J}{\delta\rho(x)}\right)^2 = \frac{1}{8}\int dx\rho(x)\left[(\lambda-1)\frac{\partial_x\rho(x)}{\rho(x)} + 2\lambda⨍dy\frac{\rho(y)}{x-y}\right]^2. \tag{8}$$

This potential is positive semidefinite and, therefore, its contribution to the ground-state energy vanishes if there exists a positive solution of the first-order differential Bogomol'nyi-type equation:

$$(\lambda - 1)\frac{\partial_x \rho(x)}{\rho(x)} + 2\lambda \fint dy \frac{\rho(y)}{x-y} = 0. \tag{9}$$

The most obvious solution is given by the constant-density configuration $\rho = \rho_0$ for any value of the statistical parametar λ. Let us now find an interesting family of ground-state solutions which represents hole excitations of the Calogero system. Using the identity for the principal distribution

$$\frac{P}{x-y}\frac{P}{x-z} + \frac{P}{y-x}\frac{P}{y-z} + \frac{P}{z-x}\frac{P}{z-y} = \pi^2 \delta(x-y)\delta(x-z), \tag{10}$$

and performing partial integration, we can rewrite V_{eff} as

$$\begin{aligned} V_{eff} = & \frac{1}{8}\int dx \rho(x) \left[(\lambda - 1)\frac{\partial_x \rho(x)}{\rho(x)} + \frac{2c}{x} + 2\lambda \fint dy \frac{\rho(y)}{x-y} \right]^2 - \\ & -\frac{1}{2}c(c-1+\lambda)\int dx \frac{\rho(x)}{x^2} + \frac{c\lambda}{2}\left(\int dx \frac{\rho(x)}{x} \right)^2 - \\ & -\frac{c\lambda}{2}\pi^2 \rho^2(0). \end{aligned} \tag{11}$$

We are looking for the symmetric minimum $\rho(x) = \rho(-x)$, representing a hole located at the origin $\rho(0) = 0$. For the particular value of the constant c given by

$$c = 1 - \lambda, \tag{12}$$

the Bogomol'nyi limit appears. The contribution of V_{eff} vanishes and the corresponding configuration satisfies the Bogomol'nyi equation

$$(\lambda - 1)\frac{\partial_x \rho(x)}{\rho(x)} + \frac{2(1-\lambda)}{x} + 2\lambda \fint dy \frac{\rho(y)}{x-y} = 0. \tag{13}$$

We now see that the role of the new singular term in the equation is to compensate for the singularity produced by $\partial_x \ln \rho(x)$ at the point where the collective field ρ vanishes. Equation (13) can be solved by a rational ansatz

$$\rho(x) = \frac{ax^2}{b^2 + x^2}. \tag{14}$$

Inserting the ansatz for $\rho(x)$ (14) into our equation (13), using the Hilbert transform

$$⨍ \frac{dy}{x-y} \frac{1}{a^2+y^2} = \frac{\pi}{a} \frac{x}{a^2+x^2}, \tag{15}$$

and after performing some calculation we find the condition

$$ab\pi = \frac{1-\lambda}{\lambda}. \tag{16}$$

An acceptable positive solution exists only for $\lambda < 1$. For large values of x, the soliton solution (14) approaches to the constant solution found before. It can be shown that the net particle number carried by the our soliton $\rho(x)$ is

$$\int dx(\rho(x) - \rho_0) = \frac{\lambda - 1}{\lambda} < 0. \tag{17}$$

This fact indicates again that our soliton corresponds to the hole excitation. There is a correspondnig prefactor in front of the wave function which describes the hole. Generally, adding a term inside the bracket (as was done in (11)), means adding a new term to the logarithm of the Jacobian. The corresponding prefactor, which arises because of adding $(1-\lambda)/(x-X)$, is $\prod_i (x_i - X)^{1-\lambda}$, describing the hole at place X.

So far we have considered soliton solutions originating from the Bogomol'nyi lower bound on the ground-state energy. Let us now turn our attention to the possible solutions which cannot be reached by the Bogomol'nyi saturation. To find such a solution, it is necessary to minimize the collective Hamiltonian with respect to ρ i π, i.e. to find the corresponding dynamical equations of motion. In this case, some interesting investigations have already been done in the recent literature [6]. We shall rederive the solution studied by the author of ref. [6], to obtain a general description, valid for any value of the statistical parametar λ.

The equations to be solved are

$$\dot{\rho} = -\partial_x(\rho(x)\partial_x \pi), \tag{18a}$$

$$\dot{\pi} = -\frac{1}{2}(\partial_x \pi)^2 - \frac{\delta V_{eff}}{\delta \rho}, \tag{18b}$$

where V_{eff} is given by (8). Since we are looking for constant-profile solutions, propagating at speed v, depending only on $\xi = x - vt$, we obtain

$$\frac{d\pi}{d\xi} = v\left(1 - \frac{\rho_0}{\rho}\right), \tag{19a}$$

$$v^2\left(1 - \frac{\rho_0}{\rho}\right) = \frac{v^2}{2}\left(1 - \frac{\rho_0}{\rho}\right)^2 + \frac{\delta V_{eff}}{\delta\rho}, \tag{19b}$$

where ρ_0 is a constant solution defined by

$$\frac{\pi^2\lambda^2}{2}\rho_0^2 = \mu. \tag{20}$$

Here, we have implemented the Lagrange multiplier μ, which defines the energy scale of the problem. Some algebra finally leads to the equation for a moving soliton-solution:

$$\begin{aligned}\frac{v^2}{2}\left(\frac{\rho_0^2}{\rho^2} - 1\right) + \frac{\pi^2\lambda^2}{2}(\rho^2 - \rho_0^2) - \frac{(\lambda-1)^2}{4}\partial_\xi\left(\frac{\partial_\xi\rho}{\rho}\right) - \\ -\frac{(\lambda-1)^2}{8}\left(\frac{\partial_\xi\rho}{\rho}\right)^2 - \lambda(\lambda-1)\partial_\xi ⨍ d\eta\frac{\rho(\eta)}{\xi-\eta} = 0.\end{aligned} \tag{21}$$

Plugging the rational ansatz for $\rho(\xi)$ into (21):

$$\rho(\xi) = \frac{\rho_0\xi^2 + a^2}{\xi^2 + b^2}, \tag{22}$$

we find the following condition on the parametars a and b:

$$b = \frac{\lambda(\lambda-1)\pi\rho_0}{v^2 - \rho_0^2\pi^2\lambda^2}, \tag{23a}$$

$$a^2 = b^2\rho_0 + \frac{\lambda-1}{\lambda\pi}b = \frac{(\lambda-1)^2v^2\rho_0}{[v^2 - \rho_0^2\pi^2\lambda^2]^2}. \tag{23b}$$

Notice that the above formulae are valid only for $b > 0$. Writing the solution $\rho(\xi)$ in the form

$$\rho(\xi) = \rho_0 + \frac{a^2 - \rho_0 b^2}{\xi^2 + b^2}, \tag{24}$$

it can be easily seen that depending on the sign of the numerator $a^2 - \rho_0 b^2$, two basically different soliton profiles emerge. For $a^2 - \rho_0 b^2 < 0$, $(\lambda < 1)$, we have a hole excitation

propagating at speed $|v| < \pi\rho_0\lambda$. For $a^2 - \rho_0 b^2 > 0$, ($\lambda > 1$), we have a lump solution propagating at speed $|v| > \pi\rho_0\lambda$. The static ($v = 0$) soliton exists only for $\lambda < 1$, and corresponds to the solution found before (14). The net particle number carried by the soliton $\rho(\xi)$ is defined by

$$\Delta N = \int d\xi(\rho(\xi) - \rho_0) = (a^2 - \rho_0 b^2) \int \frac{d\xi}{\xi^2 + b^2} = \frac{\lambda - 1}{\lambda}. \tag{25}$$

The energy of the moving soliton is defined with respect to the stationary background given by ρ_0

$$E = H(\rho(\xi)) - H(\rho_0). \tag{26}$$

Using the equation of motion (21), we obtain

$$E = \frac{\lambda - 1}{\lambda}\left(\frac{v^2}{2} - \mu\right). \tag{27}$$

Let us now find the momentum of the moving soliton. The collective-field theory gives

$$P = \sum_i p_i = \int dx \partial_x \rho(x)\pi(x). \tag{28}$$

After partial integration, the momentum (28) takes the form

$$P = -\int dx \rho(x)\partial_x \pi + \rho_0(\pi(\infty) - \pi(-\infty)). \tag{29}$$

Inserting the expression (19a) for $\partial_x \pi$, we easily obtain

$$P = \frac{\lambda - 1}{\lambda}(\sqrt{2\mu} - v). \tag{30}$$

From the relations (27) and (30), we have the following dispersion law for the moving soliton:

$$E(P) = \frac{\lambda}{\lambda - 1}\frac{P^2}{2} + \sqrt{2\mu}|P|. \tag{31}$$

We conclude by noting that our moving-soliton solution exists only for λ different from zero or one, i.e. for generic intermediate statistics. The effect of including quantum fluctuations around the moving solitons and its implications on the dispersion law will be considered elsewhere.

Note added. After this work was completed, we became aware of ref. [8]. The moving-soliton solution found there corresponds to our solution (24) only for $\lambda > 1$.

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